Exercise Sheet Solutions #3

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P1. **(Problem 3.3.)**(Borel–Cantelli lemma) If $(A_n)_{n\in\mathbb{N}}$ is a family of measurable subsets of a probability space $(X, \mathcal{B}, \mu)$ and $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$, then

$$\mu(\{x \in X \mid x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}) = 0.$$

> ***Solution:*** Notice that what we want to prove is $\mu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_j) = 0$. As μ is finite and $(\bigcup_{n\geq N} A_n)_N$ is a nested sequence of sets, we have that by continuity of the measure and subadditivity:
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> $$\mu(\bigcap_{N\in\mathbb{N}} \bigcup_{n\geq N} A_j) = \lim_{N\to\infty} \mu(\bigcup_{n\geq N} A_j) \leq \lim_{N\to\infty} \sum_{j\geq N} \mu(A_j) = 0, \tag{1}$$
>
> in where the last equality comes from the fact that $\sum_{n\in\mathbb{N}} \mu(A_n) < \infty$.

P2. Let $(X, \mathcal{F}, \mu)$ be a measure space and f a measurable function. Prove the Markov-Chebyshev inequality:

$$\forall \alpha > 0, \mu(\{|f| > \alpha\}) \leq \frac{1}{\alpha} \int_{\{|f|>\alpha\}} |f|\, d\mu \leq \frac{1}{\alpha} \int |f|\, d\mu,$$

where we denote $\{|f| > \alpha\} = \{x \in X \mid |f(x)| > \alpha\}$.

> ***Solution:*** Observe that
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> $$\mu(\{|f| > \alpha\}) = \int 1\cdot\mathbb{1}_{\{|f|>\alpha\}}(x) d\mu(x) \leq \int \frac{|f(x)|}{\alpha} \mathbb{1}_{\{|f|>\alpha\}}(x) d\mu(x) = \frac{1}{\alpha} \int_{\{|f|>\alpha\}} |f|\, d\mu \leq \frac{1}{\alpha} \int |f|\, d\mu. \tag{2}$$

P3. Let $(X, \mathcal{F}_X, \mu)$ and $(Y, \mathcal{F}_Y, \nu)$ be probability spaces, and let $T : X \to Y$ be a measurable function. Define $T\mu(A) := \mu(T^{-1}(A))$ for each $A \in \mathcal{F}_Y$. Prove that $\nu = T\mu$ if and only if for all integrable function f:

$$\int_Y f\, d\nu = \int_X f \circ T\, d\mu. \tag{3}$$

> ***Solution:*** $(\Leftarrow)$ For any set $A \in \mathcal{F}_Y$, take $f = \mathbb{1}_A$ in eq. (7) to get $\nu(A) = \mu(T^{-1}A)$.
> $(\Rightarrow)$ Let $f = \sum_{i\in I} c_i \mathbb{1}_{A_i}$ a simple function. Then
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> $$\int_X f\circ T d\mu = \int_X \sum_{i\in I} c_i \mathbb{1}_{A_i} \circ T d\mu = \sum_{i\in I} c_i \int \mathbb{1}_{T^{-1}A_i} d\mu = \sum_{i\in I} c_i \mu(T^{-1}A_i) = \sum_{i\in I} c_i \nu(A_i) = \int \sum_{i\in I} c_i \mathbb{1}_{A_i} d\nu. \tag{4}$$
>
> Hence, the statement is true for simple functions. Let f be a positive measurable function. Take $(f_n)_n$ a sequence of positive simple functions such that $f_n \nearrow f$ as $n \to \infty$. Then $f_n \circ T \nearrow f \circ T$ as $n \to \infty$ as well, so by monotone convergence theorem, we have that having for each $n \in \mathbb{N}$
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> $$\int_Y f_n\, d\nu = \int_X f_n \circ T\, d\mu, \tag{5}$$
>
> implies that
>
> $$\int_Y f\, d\nu = \lim_{n\to\infty} \int_Y f_n\, d\nu = \lim_{n\to\infty} \int_X f_n \circ T\, d\mu = \int_X f \circ T\, d\mu. \tag{6}$$

Consequently, we have the statement for positive functions. To conclude, notice that for every measurable function f, we can write $f = f_+ - f_-$ where f_+ and f_- are positive integrable functions. Thus,

$$\int_Y f\,d\nu = \int_Y f_+\,d\nu - \int_Y f_-\,d\nu = \int_X f_+ \circ T\,d\mu - \int_Y f_-\,d\nu = \int_X f_- \circ T\,d\mu = \int_X (f_+ - f_-)\circ T\,d\mu = \int_X f \circ T\,d\mu, \tag{7}$$

concluding.

P4. **(Problem 3.2.)** Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(A_n)_{n\in\mathbb{N}}$ be a family of measurable sets with $a = inf_{n\in\mathbb{N}}\mu(A_n) > 0$. We aim to show that there is a set $E \subseteq \mathbb{N}$ such that $\bar{d}(E) := \limsup_{N\to\infty} \frac{|E\cap\{1,\ldots,N\}|}{N} \geq a$, and for any finite set $F \subseteq E, F \neq \emptyset$, one has $\mu\left(\bigcap_{n\in F} A_n\right) > 0$.

Solution: First of all, notice that we can assume without loss of generality that for every $F \subseteq \mathbb{N}$ finite, we have that $\bigcap_{n\in F} A_n \neq \emptyset$ if and only if $\mu(\bigcap_{n\in F} A_n) > 0$. Indeed, we can replace $(A_n)_{n\in\mathbb{N}}$ by $(\tilde{A}_n)_{n\in\mathbb{N}}$ where

$$\tilde{A}_n = A_n \setminus \bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} A_n,$$

where

$$\mathcal{F} = \{F \subseteq \mathbb{N} \mid |F| < \infty, \bigcap_{n\in F} A_n \neq \emptyset, \mu(\bigcap_{n\in F} A_n) = 0\}.$$

Notice that $\bigcup_{F\in\mathcal{F}} \bigcap_{n\in F} A_n$ is countable union of finite intersection of measurable sets, so it is measurable. Moreover, it has measure zero because the aforementioned sets have measure zero. So, we have that for each $n \in \mathbb{N}$, $\tilde{A}_n \subseteq A_n$ is such that

$$\mu(A_n) = \mu(\tilde{A}_n),$$

and the same applies for every finite intersection, which shows that we can replace A_n by $\tilde{A}_n$ Notice that by Fatou's lemma

$$\int \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x)d\mu(x) \geq \limsup_{N\to\infty} \int \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x)d\mu(x) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A_n) \geq a.$$

Therefore, there is $x \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) \geq a.$$

Call $E = \{n \in \mathbb{N} \mid x \in A_n\}$. Then notice that $n \in E$ if and only if $\mathbb{1}_{A_n}(x) = 1$. Hence, we have that

$$\bar{d}(E) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_E(n) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{A_n}(x) \geq a.$$

In addition, for every finite subset $F \subseteq E$, we have that $\forall n \in F$, $x \in A_n$, and thus $\bigcap_{n\in F} A_n \neq \emptyset$, which implies that $\mu\left(\bigcap_{n\in F} A_n\right) > 0$.